SHEAVES ON $\mathbb{P}^1 \times \mathbb{P}^1$, BIGRADED RESOLUTIONS, AND COADJOINT ORBITS OF LOOP GROUPS

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Abstract. We construct a canonical linear resolution of acyclic 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ and discuss the resulting natural Poisson structure.

1. Introduction

The goal of this paper is to present a (yet another) variation on a theme developed by several authors, notably Moser, Adams, Harnad, Hurtubise, Previato [13], [1]–[5], and relating integrable systems, rank r perturbations, spectral curves and their Jacobians, and coadjoint orbits of loop groups.

Let us briefly recall that, given matrices A, Y, F, G of size, respectively, $N \times N$, $r \times r$, $N \times r$, and $r \times N$, one defines a $\mathfrak{gl}_r(\mathbb{C})$-valued rational map

$$Y + G(A-\lambda)^{-1}F, \tag{1.1}$$

i.e. an element of the loop algebra $\widetilde{\mathfrak{gl}}(r)^-$, consisting of loops extending holomorphically to the outside of some circle $S^1 \subset \mathbb{C}$. This determines a (shifted) reduced coadjoint orbit in $\widetilde{\mathfrak{gl}}(r)^-$ (see Remark 4.5 for a definition). On the other hand, the polynomial (1.1) also determines (generically) a curve S and a line bundle L of degree $g+r-1$: the curve is defined as the spectrum of (1.1), and L is the dual of the eigenbundle of (1.1). This describes S as an affine curve in $\mathbb{C}^2$, and the isospectral flows, corresponding to Hamiltonians on the space of rank r perturbations, linearise on the Jacobian of the projective model of S.

In fact, as shown by Adams, Harnad, and Hurtubise [1, 2], it is more convenient to compactify S inside a Hirzebruch surface F_d, $d \geq 1$. This results in singularities, which may be partially resolved, but it gives a particularly nice description of $\mathrm{Jac}^0(S)$, i.e. of the flow directions.

In this paper, we consider a different compactification of S, namely inside $\mathbb{P}^1 \times \mathbb{P}^1$ and defined as

$$S = \left\{ (z,\lambda) \in \mathbb{P}^1 \times \mathbb{P}^1;\ \det \begin{pmatrix} Y - z & G \\ F & A - \lambda \end{pmatrix} = 0 \right\}. \tag{1.2}$$

This is a very natural thing to do, but we know of only one occurence in the literature: the paper of Sanguinetti and Woodhouse [17] (we are grateful to Philip Boalch for this reference). In that paper, in addition to other results, the authors use the above compactification to give a nice picture of the duality phenomenon discussed in [3]. Our application is to another subtlety of the rank r perturbation isospectral flow: the fact that the flow may leave the set where $\operatorname{rank} F = \operatorname{rank} G = r$, without becoming singular. More precisely, we have:

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Theorem 1.1. *Let S be a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$, defined by* (1.2) *and corresponding to a (shifted) rank r perturbation of the matrix A ($r \leq N$). A line bundle $L \in \operatorname{Jac}^{g-r+1}(S)$ corresponds to (A, Y, F, G) with $\operatorname{rank} F = \operatorname{rank} G = r$ if and only if L satisfies:*

$$H^0(S, L(0,-1)) = H^1(S, L(0,-1)) = 0, \;\; H^0(S, L(-1,0)) = 0, \;\; H^1(S, L(1,-2)) = 0.$$

We are interested in more than line bundles on smooth curves in $\mathbb{P}^1 \times \mathbb{P}^1$. The above approach generalises to acyclic (i.e. semistable) 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, with a fixed bigraded Hilbert polynomial. In Sections 2 and 3, we construct a natural linear resolution of such a sheaf, very much in the spirit of Beauville [6]. This gives us a linear polynomial matrix $M(z,\lambda)$ (up to certain group action). If the support of the sheaf is a smooth curve of bidegree (r,N), then the matrix has size $r \times N$. As long as the point (∞,∞) does not belong to the support of the sheaf, then matrices $M(z,\lambda)$ can be identified with the quadruples A, Y, F, G. The space $\mathcal{M}(k,l)$ of the (A,Y,F,G) has a natural Poisson structure, obtained by identifying it with $\mathfrak{gl}_N(\mathbb{C})^* \oplus \mathfrak{gl}_r(\mathbb{C})^* \oplus T^*M_{N\times r}(\mathbb{C})$. Thus we obtain a Poisson structure on the quotient of an open subset of $\mathcal{M}(N,r)$ by $GL_N(\mathbb{C}) \times GL_r(\mathbb{C})$. The (generic) symplectic leaves are known, from [5, 1], to be reduced coadjoint orbits of loop groups. Our aim is to describe these symplectic leaves directly in terms of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. We show that they correspond to symplectic leaves of a particular Mukai-Tyurin-Bottacin Poisson structure [14, 18, 8, 9, 10, 11] on the moduli space $M_Q(r,N)$ of simple sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with (bigraded) Hilbert polynomial $Nx+ry$. The surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$ is an example of a *Poisson surface* [8], and consequently, for every choice of a Poisson structure on Q, i.e. a section s of the anticanonical bundle $K_Q^* \simeq \mathcal{O}(2,2)$, one obtains a Poisson structure on $M_Q(r,N)$ as a map

$$T^*_{[\mathcal{F}]}M_Q(r,N) \simeq \operatorname{Ext}^1_Q(\mathcal{F}, \mathcal{F}\otimes K_Q) \xrightarrow{\cdot s} \operatorname{Ext}^1_Q(\mathcal{F},\mathcal{F}) \simeq T_{[\mathcal{F}]}M_Q(r,N).$$

We show that the (generic) symplectic leaves $\mathfrak{gl}_N(\mathbb{C})^* \oplus \mathfrak{gl}_r(\mathbb{C})^* \oplus T^*M_{N\times r}(\mathbb{C})$, i.e. reduced coadjoint orbits in $\widetilde{\mathfrak{gl}}(r)^-$, are the symplectic leaves of the Mukai-Tyurin-Bottacin structure corresponding to $s(z,\lambda) = 1$, i.e. to the anticanonical divisor $2\big(\{\infty\}\times\mathbb{P}^1 + \mathbb{P}^1 \times \{\infty\}\big)$.

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2. Acyclic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ and their resolutions

Definition 2.1. Let X be a complex manifold and let $\mathcal{F}$ be a coherent sheaf on X.

(i) The *support* of $\mathcal{F}$ is the complex subspace $\operatorname{supp}\mathcal{F}$ of X defined as the zero-locus of the annihilator (in $\mathcal{O}_X$) of $\mathcal{F}$. The dimension $\dim \mathcal{F}$ of $\mathcal{F}$ is the dimension of its support.

(ii) $\mathcal{F}$ is *pure*, if $\dim \mathcal{E} = \dim \mathcal{F}$ for all non-trivial coherent subsheaves $\mathcal{E} \subset \mathcal{F}$.

(iii) $\mathcal{F}$ is *acyclic* if $H^*(\mathcal{F}) = 0$.

Remark 2.2. In the case of 1-dimensional sheaves on a smooth surface X, purity of $\mathcal{F}$ means that, at at every point $x \in \operatorname{supp}\mathcal{F}$, the skyscraper sheaf $\mathbb{C}_x$ does not embed into $\mathcal{F}_x$. In addition, a 1-dimensional sheaf $\mathcal{F}$ on a smooth surface X is pure if and only if it is *reflexive*, i.e. after performing the duality $\mathcal{F} \mapsto \mathcal{E}xt^1_X(\mathcal{F}, K_X)$ twice, we obtain back $\mathcal{F}$ (up to isomorphism) (see [9, §1.1]).

In the remainder of the paper, **all sheaves are coherent**.

We shall now consider sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. For any $p, q \in \mathbb{Z}$ we denote by $\mathcal{O}(p,q)$ the line bundle $\pi_1^*\mathcal{O}(p) \otimes \pi_2^*\mathcal{O}(q)$, where $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ are the two projections. We shall also denote by ζ and η the two affine coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. Associated to $\mathcal{F}$ is its *bigraded Hilbert polynomial*

$$P_{\mathcal{F}}(x,y) = \sum_{x,y\in\mathbb{Z}} \chi(\mathcal{F}(x,y)). \tag{2.1}$$

The sheaf $\mathcal{F}$ is 1-dimensional if and only if $P_{\mathcal{F}}$ is linear.

We begin by describing a canonical resolution of acyclic 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 2.3. *Let $\mathcal{F}$ be a 1-dimensional acyclic sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathcal{F}$ has a linear resolution by locally free sheaves of the form*

$$0 \to \mathcal{O}(-2,-1)^{\oplus k} \oplus \mathcal{O}(-1,-2)^{\oplus l} \xrightarrow{M(\zeta,\eta)} \mathcal{O}(-1,-1)^{\oplus(k+l)} \to \mathcal{F} \to 0, \tag{2.2}$$

for some $k, l \geq 0$.

Conversely, any $\mathcal{F}$ defined as cokernel of a map $M(\zeta,\eta)$ as above with $\det M(\zeta,\eta) \not\equiv 0$ *is acyclic and* 1*-dimensional.*

Remark 2.4. Let $\mathcal{F}$ be a 1-dimensional acyclic sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with $P_{\mathcal{F}}(x,y) = lx + ky$. Then $\mathcal{F}$ is semistable with respect to $\mathcal{O}(1,1)$.

Remark 2.5. This resolution is canonical, but not necessarily *minimal*, in the sense of being obtained from the minimal resolution of the bigraded module $\bigoplus_{i,j\in\mathbb{Z}} H^0(\mathcal{F}(i,j))$.

Proof. Let $h^0(\mathcal{F}(0,1)) = k$ and $h^0(\mathcal{F}(1,0)) = l$, so that $P_{\mathcal{F}} = lx + ky$. Let $\mathcal{E} = \mathcal{F}(1,1)$, and let $\mathbf{\Gamma}_*(\mathcal{E}) = \bigoplus_{i,j\in\mathbb{Z}} H^0(\mathcal{E}(i,j))$ be the associated bigraded module over the bigraded ring $\mathbf{S} = \bigoplus_{i,j\in\mathbb{Z}} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(i,j))$. Furthermore, let $\mathbf{\Gamma}_*(\mathcal{E})|_{\geq 0} = \bigoplus_{i,j\geq 0} H^0(\mathcal{E}(i,j))$ be its truncation. Owing to [12, Lemma 6.8], the sheaf associated to $\mathbf{\Gamma}_*(\mathcal{E})|_{\geq 0}$ is again $\mathcal{E}$. Moreover, [12, Theorem 6.9] implies, as $\mathcal{E}(-1,-1)$ is acyclic, that the natural map

$$H^0(\mathcal{E}) \otimes H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(p,q)) \longrightarrow H^0(\mathcal{E}(p,q))$$

is surjective for any $p, q \geq 0$. Therefore, we have a surjective homomorphism

$$\mathbf{S}^{\oplus(k+l)} \to \mathbf{\Gamma}_*(\mathcal{E})|_{\geq 0} \to 0$$

of bigraded $\mathbf{S}$-modules. Since $\mathcal{E}$ is of pure dimension 1, its projective dimension is 1, and, hence, the above homomorphism extends to a linear free resolution

$$0 \to \bigoplus_{i=1}^{k+l} \mathbf{S}(-p_i,-q_i) \to \bigoplus_{i=1}^{k+l} \mathbf{S} \to \mathbf{\Gamma}_*(\mathcal{E})|_{\geq 0} \to 0,$$

where $p_i, q_i \geq 0$ and $p_i + q_i > 0$ for each i. The corresponding sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ give us a locally free resolution of $\mathcal{E}$:

$$0 \to \bigoplus_{i=1}^{k+l} \mathcal{O}(-p_i,-q_i) \to \bigoplus_{i=1}^{k+l} \mathcal{O} \to \mathcal{E} \to 0. \tag{2.3}$$

Since $H^*(\mathcal{E}(-1,-1)) = 0$, either $p_i = 0$ or $q_i = 0$ for every i. Since $h^0(\mathcal{E}(-1,0)) = k$, we deduce, after tensoring (2.3) with $\mathcal{O}(-1,0)$, that $\sum p_i = k$. Similarly $\sum q_i =$

l. Since $h^1(\mathcal{E})=0$, none of the p_i or q_i can be greater than 1, and so, all nonzero p_i and all nonzero q_i are equal to 1. This proves the existence of resolution (2.2).

Conversely, if $\mathcal{F}$ admits a resolution of the form (2.2), then it is 1-dimensional. The long exact cohomology sequence implies that $\mathcal{F}$ is acyclic. □

Let us write $n=k+l$. The polynomial matrix $M(\zeta,\eta)$ in (2.3) has size $n\times n$ and is of the form

$$\begin{pmatrix} A_0+A_1\zeta & B_0+B_1\eta \end{pmatrix}, \tag{2.4}$$

with $A_0,A_1\in \mathrm{Mat}_{n,k}(\mathbb{C})$, $B_0,B_1\in \mathrm{Mat}_{n,l}(\mathbb{C})$. Let us denote by $\mathcal{A}(k,l)$ the space of such matrices with nonzero determinant. The group $GL_n(\mathbb{C})\times GL_k(\mathbb{C})\times GL_l(\mathbb{C})$ acts on $\mathcal{M}(k,l)$ via:

$$(g,h_1,h_2).\begin{pmatrix} A(\zeta) & B(\eta)\end{pmatrix} = g\begin{pmatrix} A(\zeta) & B(\eta)\end{pmatrix}\begin{pmatrix} h_1^{-1} & 0\\ 0 & h_2^{-1}\end{pmatrix}, \tag{2.5}$$

and we can restate Theorem 2.3 as follows:

Corollary 2.6. *There exists a natural bijection between*

(a) *isomorphism classes of* 1*-dimensional acyclic sheaves* $\mathcal{F}$ *on* $\mathbb{P}^1\times\mathbb{P}^1$ *such that* $h^0(\mathcal{F}(0,1))=k$, $h^0(\mathcal{F}(1,0))=l$,
and
(b) *orbits of* $GL_{k+l}(\mathbb{C})\times GL_k(\mathbb{C})\times GL_l(\mathbb{C})$ *on* $\mathcal{A}(k,l)$. □

For a sheaf define by (2.2), we can describe its support as follows. As a set, the support of $\mathcal{F}$ is

$$S=\{(\zeta,\eta)\in\mathbb{P}^1\times\mathbb{P}^1;\ \det M(\zeta,\eta)=0\}.$$

Let us write $\det M(\zeta,\eta)=\prod_{i=1}^s q_i(\zeta,\eta)^{k_i}$, where q_i are irreducible polynomials. We define the *minimal polynomial* $p_M(\zeta,\eta)$ of M as $\prod_{i=1}^s q_i(\zeta,\eta)^{r_i}$, where

$$\begin{aligned} r_i \;=\; & \max\{a_ib_i;\ \text{at a generic point, } M(\zeta,\eta) \text{ has a Jordan block of size } a_i \\ & \text{with eigenvalue } q_i(\zeta,\eta)^{b_i}\}. \end{aligned}$$

Then:

Proposition 2.7. *The support of* $\mathcal{F}$ *is the curve* $\left(S,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}/(p_M)\right)$. □

Let us now fix the support S. For simplicity, we shall assume that it is an *integral* curve in the linear system $|\mathcal{O}(k,l)|$ on $\mathbb{P}^1\times\mathbb{P}^1$, i.e. S is given by an irreducible polynomial $P(\zeta,\eta)$ of bidegree (k,l), $k,l\geq 1$. This immediately implies that the rank of $\mathcal{F}$ is constant, i.e. $\mathcal{F}$ is locally free. Theorem 2.3 and Corollary 2.6 imply

Corollary 2.8. *Let* $P(\zeta,\eta)$ *be an irreducible polynomial of bidegree* (k,l)*, and* $S=\{(\zeta,\eta);\ P(\zeta,\eta)=0\}$ *the corresponding integral curve of genus* $g=(k-1)(l-1)$*. There exists a canonical biholomorphism*

$$\mathrm{Jac}^{g-1}(S)-\Theta\simeq\{M\in\mathcal{A}(k,l);\ \det M=P\}/GL_n(\mathbb{C})\times GL_k(\mathbb{C})\times GL_l(\mathbb{C}).$$

Similarly, let $\mathcal{U}_S(r,d)$ be the moduli space of semistable vector bundles (locally free sheaves) on S. For $d=r(g-1)$ define the generalised theta divisor Θ as the set of bundles with nonzero section. Then we have:

Corollary 2.9. *Let* $P(\zeta,\eta)$ *be an irreducible polynomial of bidegree* (k,l)*, and* $S=\{(\zeta,\eta);\ P(\zeta,\eta)=0\}$ *the corresponding integral curve of genus* $g=(k-1)(l-1)$*. There exists a canonical biholomorphism*

$$\mathcal{U}_S(r,r(g-1))-\Theta\simeq\{M\in\mathcal{A}(kr,lr);\ \det M=P^r\}/GL_{nr}(\mathbb{C})\times GL_{kr}(\mathbb{C})\times GL_{lr}(\mathbb{C}).$$

3. A geometric resolution

There is a much more geometric way of constructing resolution (2.2), which works under mild assumptions on the sheaf $\mathcal{F}$ (cf. [7] for the case of σ-sheaves).

Definition 3.1. Let $\mathcal{F}$ be a 1-dimensional sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ and let $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the two projections. We say that $\mathcal{F}$ is *bipure*, if $\mathcal{F}$ has no nontrivial coherent subsheaves supported on $\{z\} \times \mathbb{P}^1$ or on $\mathbb{P}^1 \times \{z\}$ for any $z \in \mathbb{P}^1$.

Let now $\mathcal{F}$ be an acyclic and bipure sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $lx + ky$. As in the proof of Theorem 2.3, we consider the sheaf $\mathcal{E} = \mathcal{F}(1,1)$. Let D_ζ and D_η denote the divisors $\{\zeta\} \times \mathbb{P}^1$, $\mathbb{P}^1 \times \{\eta\}$. We set

$$V_\zeta = \{s \in H^0(\mathcal{E}); s|_{D_\zeta} = 0\}, \quad W_\eta = \{s \in H^0(\mathcal{E}); s|_{D_\eta} = 0\}. \tag{3.1}$$

For any ζ and η, consider the maps

$$\mathcal{E}(-1,0) \to \mathcal{E}, \quad \mathcal{E}(0,-1) \to \mathcal{E},$$

given by multiplication by global non-zero sections of $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$, vanishing at ζ and η, respectively. Since $\mathcal{E}$ is bipure, these maps are injective, and therefore $V_\zeta \simeq H^0(\mathcal{E}(-1,0))$, $W_\eta \simeq H^0(\mathcal{E}(0,-1))$ for any ζ, η. In particular $\dim V_\zeta = k$, $\dim W_\eta = l$, for any ζ and η. Therefore, $\zeta \mapsto V_\zeta$ and $\eta \mapsto W_\eta$ are subbundles of $H^0(\mathcal{E}) \otimes \mathcal{O}$ on $\mathbb{P}^1$. They are isomorphic to $H^0(\mathcal{E}(-1,0)) \otimes \mathcal{O}(-1)$, and to $H^0(\mathcal{E}(0,-1)) \otimes \mathcal{O}(-1)$. The isomorphism is realised explicitly via the map: $H^0(\mathcal{E}(-1,0)) \otimes \mathcal{O}(-1) \to H^0(\mathcal{E}) \otimes \mathcal{O}$, defined as

$$H^0(\mathcal{E}(-1,0)) \otimes \mathcal{O}(-1) \ni (s,(a,b)) \overset{m}{\longmapsto} (b\zeta - a)s \in H^0(\mathcal{E})$$

(here $(a,b) \in l$, where l is the fibre of $\mathcal{O}(-1)$ over $[l]$), and similarly for the subbundle W. We now define a vector bundle U on $\mathbb{P}^1 \times \mathbb{P}^1$, the fibre of which at $\zeta, \eta)$ is $V_\zeta \oplus W_\eta$, i.e.:

$$U \simeq \left(H^0(\mathcal{E}(-1,0)) \otimes \mathcal{O}(-1,0)\right) \oplus \left(H^0(\mathcal{E}(0,-1)) \otimes \mathcal{O}(0,-1)\right).$$

We obtain an injective map of sheaves $\mathcal{U} \to H^0(\mathcal{E}) \otimes \mathcal{O}$. Let $\mathcal{G}$ be the cokernel, i.e.

$$0 \to \mathcal{U} \longrightarrow H^0(\mathcal{E}) \otimes \mathcal{O} \longrightarrow \mathcal{G} \to 0. \tag{3.2}$$

We claim that $\mathcal{G} \simeq \mathcal{E}$, and so (3.2) is a natural resolution of $\mathcal{E}$. To prove this, tensor the resolution (2.2) by $\mathcal{O}(1,1)$ to obtain:

$$0 \to \mathcal{O}(-1,0)^{\oplus k} \oplus \mathcal{O}(0,-1)^{\oplus l} \overset{M(\zeta,\eta)}{\longrightarrow} \mathcal{O}^{\oplus(k+l)} \to \mathcal{E} \to 0. \tag{3.3}$$

Clearly, the middle term is identified with $H^0(\mathcal{E}) \otimes \mathcal{O}$. For any ζ_0, consider the image of $M(\zeta_0, \eta)$ restricted to $\mathcal{O}(-1,0)^{\oplus k}|_{\zeta_0} \oplus 0$. This image does not depend on η, and since $\mathcal{F}$ is bipure, it is exactly V_{ζ_0}, defined in (3.1), i.e. sections vanishing on $\zeta_0 \times \mathbb{P}^1$. Similarly, for any η_0, the image of $M(\zeta, \eta_0)$ restricted to $0 \oplus \mathcal{O}(0,-1)^{\oplus l}|_{\eta_0}$ is precisely W_{η_0}. Hence, there are canonical isomorphisms between both first and second terms in resolutions (3.2) and (3.3), which commute with the horizontal maps. Therefore $\mathcal{G} \simeq \mathcal{E}$.

4. Poisson structure and orbits of loop groups

According to Corollary 2.6, acyclic sheaves with Hilbert polynomial $lx + ky$ correspond to orbits of $GL_{k+l}(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\mathcal{A}(k,l)$, where $\mathcal{A}(k,l)$ is the set of polynomial matrices defined in (2.4) and the action is given in (2.5).

We now make the following assumption about the sheaf $\mathcal{F}$:

$$(\infty,\infty)\notin \operatorname{supp}\mathcal{F}. \tag{4.1}$$

This can be, of course, always achieved via an automorphism of $\mathbb{P}^1\times\mathbb{P}^1$. In terms of the matrix $M(\zeta,\eta)$ corresponding to $\mathcal{F}$, (4.1) means that $\det(A_1,B_1)\neq 0$. We can, therefore, use the action of $GL_{k+l}(\mathbb{C})$ to make (A_1,B_1) equal to minus the identity matrix, so that $M(\zeta,\eta)$ becomes

$$\begin{pmatrix} X-\zeta & F\\ G & Y-\eta\end{pmatrix},\;\; X\in \mathrm{Mat}_{k,k}(\mathbb{C}),\, Y\in\mathrm{Mat}_{l,l}(\mathbb{C}),\, G,F^T\in\mathrm{Mat}_{l,k}(\mathbb{C}). \tag{4.2}$$

The residual group action is that of conjugation by the block-diagonal $GL_k(\mathbb{C})\times GL_l(\mathbb{C})$. We denote this group by K.

Remark 4.1. We are, essentially, in the situation of [5]. The only difference is that we do not fix X or Y.

We denote by $\mathcal{M}(k,l)$ the space of all matrices of the form (4.2), which we identify with quadruples (X,Y,F,G) as above. The action of $K=GL_k(\mathbb{C})\times GL_l(\mathbb{C})$ on $\mathcal{M}(k,l)$ is given by

$$(g,h).(X,Y,F,G)=(gXg^{-1},hYh^{-1},gFh^{-1},hGg^{-1}). \tag{4.3}$$

Let us also write $\mathcal{S}(k,l)$ for the set of isomorphism classes of acyclic sheaves with Hilbert polynomial $lx+ky$ on $\mathbb{P}^1\times\mathbb{P}^1$, which satisfy (4.1). The content of Corollary 2.6 is that there exists a natural bijection

$$\mathcal{M}(k,l)/K\simeq \mathcal{S}(k,l). \tag{4.4}$$

4.1. **Poisson structure.** The vector space $\mathrm{Mat}_{k,l}\times\mathrm{Mat}_{l,k}$ has a natural K-invariant symplectic structure: $\omega=\operatorname{tr}(dF\wedge dG)$. On the other hand, $\mathrm{Mat}_{k,k}\simeq\mathfrak{gl}_k(\mathbb{C})^*$ and $\mathrm{Mat}_{l,l}\simeq\mathfrak{gl}_l(\mathbb{C})^*$ have canonical Poisson structures, and therefore, $\mathcal{M}(k,l)$ has a natural K-invariant Poisson structure. If $\mathcal{M}(k,l)^0$ is the subset of $\mathcal{M}(k,l)$, on which the action of K is free and proper, then $\mathcal{M}(k,l)^0/K$ is a Poisson manifold, and, consequently, we obtain a Poisson structure on the corresponding subset of acyclic sheaves with Hilbert polynomial $lx+ky$ and satisfying (4.1). We shall now want to describe symplectic leaves of $\mathcal{M}(k,l)^0/K$ in terms of sheaves on $\mathbb{P}^1\times\mathbb{P}^1$.

First of all, let us describe sheaves corresponding to symplectic leaves in $\mathcal{M}(k,l)$. Such a leaf is determined by fixing conjugacy classes of X and Y. On the other hand, conjugacy classes of $k\times k$ matrices correspond to isomorphism classes of torsion sheaves on $\mathbb{P}^1$, of length k. This correspondence is given by associating to a matrix $X\in\mathrm{Mat}_{k,k}(\mathbb{C})$ the sheaf $\mathcal{G}$ via

$$0\to\mathcal{O}(-1)^{\oplus k}\xrightarrow{X-\zeta}\mathcal{O}^{\oplus k}\to\mathcal{G}\to 0. \tag{4.5}$$

If, for example, X is diagonalisable with distinct eigenvalues $\zeta_1,\dots,\zeta_r$ of multiplicities $k_1,\dots,k_r$, then $\mathcal{G}\simeq\bigoplus_{i=1}^r\mathbb{C}^{k_i}|_{\zeta_i}$, i.e. $\mathcal{G}|_{\zeta_i}$ is the skyscraper sheaf of rank k_i.

Proposition 4.2. *Let P be a conjugacy class of $k\times k$ matrices. The bijection* (4.4) *induces a bijection between*

(i) *orbits of $GL_k(\mathbb{C})\times GL_l(\mathbb{C})$ on $\{(X,Y,F,G)\in\mathcal{M}(k,l);\ X\in P\}$, and*
(ii) *isomorphism classes of sheaves $\mathcal{F}$ in $\mathcal{S}(k,l)$ such that $\mathcal{F}|_{\eta=\infty}$ is isomorphic to $\mathcal{G}$ defined by* (4.5).

Proof. At $\eta=\infty$, the matrix (4.2) becomes $\begin{pmatrix} X-\zeta & 0 \\ G & -1\end{pmatrix}$. The statement follows from (4.5) and (2.2). □

Therefore symplectic leaves on $\mathcal{M}(k,l)$ correspond to fixing isomorphism classes of $\mathcal{F}|_{\eta=\infty}$ and of $\mathcal{F}|_{\zeta=\infty}$. Symplectic leaves on $\mathcal{M}(k,l)^0/K$ are of course smaller than K-orbits of symplectic leaves on $\mathcal{M}(k,l)^0$. They are obtained by fixing X and Y and taking the symplectic quotient of $\mathrm{Mat}_{k,l}\times\mathrm{Mat}_{l,k}$ by $\mathrm{Stab}(X)\times\mathrm{Stab}(Y)$. We shall describe sheaves corresponding to a particular symplectic leaf in the case when X and Y are diagonalisable.

4.2. **Orbits of $GL_k(\mathbb{C})$ and matrix-valued rational maps.** We consider now only the action of $GL_k(\mathbb{C})\simeq GL_k(\mathbb{C})\times\{1\}\subset K$ on $\mathcal{M}(k,l)$. We fix a semisimple conjugacy class of X, i.e. we suppose that X is diagonalisable, with distinct eigenvalues $\zeta_1,\dots,\zeta_r$ of multiplicities $k_1,\dots,k_r$. The stabiliser of X is then isomorphic to $\prod_{i=1}^r GL_{k_i}(\mathbb{C})$. If the action of $GL_k(\mathbb{C})$ is to be free, we must have $k_i\le l$, $i=1,\dots,r$. Let us diagonalise X, so that X has the block-diagonal form $(\zeta_1\cdot 1_{k_1\times k_1},\dots,\zeta_r\cdot 1_{k_r\times k_r})$, and let F_i,G_i denote the $k_i\times l$ and $l\times k_i$ submatrices of F,G such that rows of F and the columns of G have the same coordinates as the block $\zeta_i\cdot 1_{k_i\times k_i}$. The action of $GL_k(\mathbb{C})$ is free and proper at (X,Y,F,G) if and only if $\operatorname{rank}F_i=\operatorname{rank}G_i=k_i$ for $i=1,\dots,r$.

As in [5, 1], we can associate to each element of $\mathcal{M}(k,l)$ a $\mathrm{Mat}_{l,l}(\mathbb{C})$-valued rational map:

$$R(\zeta)=Y+G(\zeta-X)^{-1}F. \tag{4.6}$$

The mapping $(X,Y,F,G)\mapsto R(\zeta)$ is clearly $GL_k(\mathbb{C})$-invariant. If X is diagonalisable, as above, i.e. $X=(\zeta_1\cdot 1_{k_1\times k_1},\dots,\zeta_r\cdot 1_{k_r\times k_r})$, then

$$R(\zeta)=Y+\sum_{i=1}^r\frac{G_iF_i}{\zeta-\zeta_i}. \tag{4.7}$$

We clearly have:

Lemma 4.3. *Let P be a semisimple conjugacy class of $k\times k$ matrices with eigenvalues $\zeta_1,\dots,\zeta_r$ of multiplicities $k_1,\dots,k_r$. The map $(X,Y,F,G)\mapsto R(\zeta)$ induces a bijection between*

(i) *$GL_k(\mathbb{C})$-orbits on $\{(X,Y,F,G)\in\mathcal{M}(k,l)^0;\ X\in P\}$, and*
(ii) *the set $\mathcal{R}_l(P)$ of all rational maps of the form*

$$R(\zeta)=Y+\sum_{i=1}^r\frac{R_i}{\zeta-\zeta_i},$$

where $\operatorname{rank}R_i=k_i$. □

4.3. **Orbits of loop groups.** A rational map of the form (4.6) may be viewed as an element of a loop Lie algebra $\widetilde{\mathfrak{gl}}(l)^-$, consisting of maps from a circle S^1 in $\mathbb{C}$, containing the points ζ_i in its interior, which extend holomorphically outside S^1 (including ∞). The group $\widetilde{GL}(l)^+$, consisting of smooth maps $g:S^1\to GL_l(\mathbb{C})$, extending holomorphically to the interior of S^1, acts on $\widetilde{\mathfrak{gl}}(l)^-$ by pointwise conjugation, followed by projection to $\widetilde{\mathfrak{gl}}(l)^-$. In particular, if all eigenvalues of X are

distinct, then the action is

$$g(\zeta).\left(Y+\sum_{i=1}^{r}\frac{R_i}{\zeta-\zeta_i}\right)=Y+\sum_{i=1}^{r}\frac{g(\zeta_i)R_ig(\zeta_i)^{-1}}{\zeta-\zeta_i}.$$

Therefore, if we fix conjugacy classes of the R_i, we obtain an orbit of $\widetilde{GL}(l)^+$ in $\widetilde{\mathfrak{gl}}(l)^-$. We shall now consider quotients of such orbits by $\mathrm{Stab}(Y)$ and describe which sheaves correspond to elements of such an orbit. Let us give a name to such quotients:

Definition 4.4. The quotient of an orbit of $\widetilde{GL}(l)^+$ in $\widetilde{\mathfrak{gl}}(l)^-$ by $GL_l(\mathbb{C})$ is called a *semi-reduced orbit*.

Remark 4.5. In the literature (see, e.g. [1]–[5]) a *reduced* orbit is the symplectic quotient of an orbit by $H_Y=\mathrm{Stab}(Y)$. The $GL_l(\mathbb{C})$-moment map on $\widetilde{\mathfrak{gl}}(l)^-$ is identified with $Y+\sum_{i=1}^r R_i$, so that a reduced orbit is obtained by fixing the value of $a=\pi\big(\sum_{i=1}^r R_i\big)$, where π is the projection $\mathfrak{gl}_l(\mathbb{C})\to\mathfrak{gl}_l(\mathbb{C})/\mathfrak{h}_Y^{\perp}$ (with $\perp$ is taken with respect to tr), and dividing by $\mathrm{Stab}(a)\subset\mathrm{Stab}(Y)$. Therefore, if $\mathrm{Stab}(Y)$ fixes a, then a reduced orbit can be identified with a subset of a semi-reduced orbit.

Let us, therefore, fix a semi-reduced orbit of $\widetilde{GL}(l)^+$. We choose r distinct points $\zeta_1,\dots,\zeta_r$ in $\mathbb{C}$. Furthermore, we choose $r+1$ conjugacy classes $Q_0,Q_1,\dots,Q_r$ of $l\times l$ matrices. This data determines a semi-reduced orbit $\Upsilon=\Upsilon(Q_0,\dots,Q_r)$ of $\widetilde{GL}(l)^+$ defined as

$$\Upsilon=\left\{R(\zeta)=Y+\sum_{i=1}^{r}\frac{R_i}{\zeta-\zeta_i};\ Y\in Q_0,\ \forall_{i\geq 1}R_i\in Q_i\right\}/GL_l(\mathbb{C}). \tag{4.8}$$

Let

$$k_i=\operatorname{rank}Q_i,\ \ i=1,\dots,r,\quad k=\sum_{i=1}^{r}k_i. \tag{4.9}$$

In the notation of Lemma 4.3, $\Upsilon\subset\mathcal{R}_l(P)$, where P is the semisimple conjugacy class of $k\times k$ matrices with eigenvalues ζ_i of multiplicities k_i.

Thanks to Proposition 4.2, the conjugacy class P determines $\mathcal{F}|_{\eta=\infty}$, which, in the case at hand, is $\bigoplus_{i=1}^r\mathbb{C}^{k_i}|_{(\zeta_i,\infty)}$. Similarly, Q_0 determines the isomorphism class of $\mathcal{F}|_{\zeta=\infty}$. We now discuss the significance of the other conjugacy classes $Q_1,\dots,Q_l$.

We claim that they determine the isomorphism class of $\mathcal{F}|_{\eta^2=\infty}$, i.e. of $\mathcal{F}$ restricted to the first order neighbourhood of $\eta=\infty$. Indeed, consider again the canonical resolution (2.2) of $\mathcal{F}$ with $M(\zeta,\eta)$ given by (4.2). Let $\tilde\eta=1/\eta$ be a local coordinate near $\eta=\infty$, so that

$$M(\zeta,\tilde\eta)=\begin{pmatrix}X-\zeta & \tilde\eta F\\ G & \tilde\eta Y-1\end{pmatrix}.$$

Using action (2.5), we can multiply $M(\zeta,\tilde\eta)$ on the right by $\begin{pmatrix}1 & 0\\ 0 & (1-\tilde\eta Y)^{-1}\end{pmatrix}$. On the scheme $\tilde\eta^2=0$, we have $(1-\tilde\eta Y)^{-1}=1+\tilde\eta Y$, and so $M(\zeta,\tilde\eta)$ becomes (on $\tilde\eta^2=0$):

$$\begin{pmatrix}X-\zeta & \tilde\eta F\\ G & -1\end{pmatrix}.$$

To describe $\mathcal{F}|_{\tilde{\eta}^2=0}$, it is enough to describe it near each ζ_i, i.e. to describe $\mathcal{G}_i = \mathcal{F}|_{U_i\times\{\tilde{\eta}^2=0\}}$, where U_i is an open neighbourhood of ζ_i (not containing the other ζ_j). The resolution (2.2) of $\mathcal{F}$ restricted to $U_i \times \{\tilde{\eta}^2 = 0\}$ becomes

$$0 \to \mathcal{O}(-2,-1)^{\oplus k_i} \oplus \mathcal{O}(-1,-2)^{\oplus l} \xrightarrow{M_i(\zeta,\tilde{\eta})} \mathcal{O}(-1,-1)^{\oplus(k_i+l)} \longrightarrow \mathcal{G}_i \to 0,$$

where

$$M_i(\zeta,\tilde{\eta}) = \begin{pmatrix} \zeta_i - \zeta & \tilde{\eta}F_i \\ G_i & -1 \end{pmatrix}.$$

This implies that we have an exact sequence

$$0 \to \mathcal{O}(-2,-1)^{\oplus k_i} \xrightarrow{(\zeta_i-\zeta)+\tilde{\eta}F_iG_i} \mathcal{O}(-1,0)^{\oplus k_i} \longrightarrow \mathcal{G}_i \to 0, \tag{4.10}$$

on $U_i \times \{\tilde{\eta}^2 = 0\}$. Therefore $\mathcal{G}_i$ is determined by the $GL_{k_i}(\mathbb{C})$-conjugacy class of F_iG_i, which is the same as the $GL_l(\mathbb{C})$-conjugacy class of G_iF_i. Lemma 4.3 and formula (4.7) imply that the conjugacy class of G_iF_i is Q_i. Thus, the conjugacy classes $Q_1,\dots,Q_r$, which determine the orbit (4.8), correspond to the isomorphism class of $\mathcal{F}|_{\tilde{\eta}^2=0}$. Observe that the support of $\mathcal{G}_i$ is given by $\det((\zeta_i-\zeta)+\tilde{\eta}F_iG_i) = 0$. In other words, the eigenvalues of F_iG_i give $\frac{\zeta-\zeta_i}{\tilde{\eta}}$ at $(\zeta,\tilde{\eta}) = (\zeta_i,0)$, i.e. the first order neighbourhood of $\operatorname{supp}\mathcal{F}$ at (ζ_i,∞).

Summing up, we have:

Theorem 4.6. *There exists a natural bijection between elements of the semi-reduced rational orbit* (4.8) *of* $\widetilde{GL}(l)^+$ *in* $\widetilde{\mathfrak{gl}}(l)^-$ *and isomorphism classes of* 1-*dimensional acyclic sheaves* $\mathcal{F}$ *on* $\mathbb{P}^1\times\mathbb{P}^1$ *such that*

(i) *the Hilbert polynomial of* $\mathcal{F}$ *is* $P_{\mathcal{F}}(x,y) = lx + ky$.
(ii) $(\infty,\infty) \notin \operatorname{supp} S$, *and* $\mathcal{F}|_{\eta=\infty} \simeq \bigoplus_{i=1}^r \mathbb{C}^{k_i}|_{(\zeta_i,\infty)}$.
(iii) *The isomorphism class of* $\mathcal{F}|_{\zeta=\infty}$ *corresponds to* Q_0, *as in Proposition 4.2.*
(iv) *The isomorphism class of* $\mathcal{F}|_{\eta^2=\infty}$ *corresponds to conjugacy classes* $Q_1,\dots,Q_r$, *as described above.* □

Remark 4.7. A variation of this result is probably well known to the integrable systems community (at least when $\mathcal{F}$ is a line bundle supported on a smooth curve S). We think it useful, however, to state it in this language and in full generality.

4.4. **Symplectic leaves of** $\mathcal{M}(k,l)^0/K$**.** We can finally describe symplectic leaves of $\mathcal{S}(k,l)$, i.e. sheaves corresponding to a particular symplectic leaf L in $\mathcal{M}(k,l)/K$, at least in the case when $L \subset \mathcal{M}(k,l)^0/K$, and X and Y are semisimple. As we already mentioned in §4.1, a symplectic leaf in $\mathcal{M}(k,l)^0/K$ is obtained by fixing X and Y, as well as a coadjoint orbit $\Lambda \subset \mathfrak{h}^*$ of $H = \operatorname{Stab}(X)\times\operatorname{Stab}(Y)$. If $\mu : \operatorname{Mat}_{k,l}\times\operatorname{Mat}_{l,k} \to \mathfrak{h}^*$ is the moment map for H, then the symplectic leaf determined by these data is:

$$L = \{(X,Y,F,G) \in \mathcal{M}(k,l)^0;\ X \text{ and } Y \text{ are given},\ \ \mu(F,G)\in\Lambda\}/H. \tag{4.11}$$

Let X be diagonal, written as in §4.2, i.e. $X = (\zeta_1\cdot 1_{k_1\times k_1},\dots,\zeta_r\cdot 1_{k_r\times k_r})$ and let F_i, G_i, $i = 1,\dots,r$, be the corresponding submatrices of F and G. Then $\operatorname{Stab}(X) \simeq \prod_{i=1}^r GL_{k_i}(\mathbb{C})$, and the moment map is the projection of the $GL_k(\mathbb{C})$-moment map, i.e. $(F,G)\mapsto FG$, onto the Lie algebra of $\operatorname{Stab}(X)$. In other words, the $\operatorname{Stab}(X)$-moment map can be identified with [5]:

$$\mu_X(F,G) = (F_1G_1,\dots,F_rG_r). \tag{4.12}$$

Similarly, if Y is diagonal with s distinct eigenvalues of multiplicities $l_1, \ldots, l_s$, then we obtain $l_i \times k$ and $k \times l_i$ submatrices G^i, F^i. The stabiliser of Y is isomorphic to $\prod_{i=1}^s GL_{l_i}(\mathbb{C})$ and the moment map is

$$\mu_Y(F, G) = (G^1F^1, \ldots, G^sF^s). \tag{4.13}$$

Therefore, an orbit Λ corresponds to $r+s$ conjugacy classes $\pi_1, \ldots, \pi_r, \rho_1, \ldots, \rho_s$ of $k_i \times k_i$ matrices for the π_i, and $l_j \times l_j$ matrices for the ρ_j. The leaf L will be contained in $\mathcal{M}(k,l)^0/K$ if and only if each conjugacy class consists of matrices of maximal rank (k_i or l_j). From the discussion in the previous subsection, we immediately obtain:

Proposition 4.8. *Let L be a symplectic leaf of the Poisson manifold $\mathcal{M}(k,l)^0/K$, defined as in* (4.11) *with semisimple X and Y. Then the image of L under the bijection* (4.4) *consists of isomorphism classes of sheaves $\mathcal{F}$ in $\mathcal{S}(k,l)$ such that the isomorphism class of $\mathcal{F}|_{\zeta^2=\infty}$ and of $\mathcal{F}|_{\eta^2=\infty}$ is fixed (and determined by L).* □

Spelling things out, X determines $\mathcal{F}|_{\eta=\infty} \simeq \bigoplus_{i=1}^r \mathbb{C}^{k_i}|_{(\zeta_i,\infty)}$, and each π_i, $i = 1, \ldots, r$, determines $\mathcal{F}$ restricted to a neighbourhood of (ζ_i, ∞) in $\{\eta^2 = \infty\}$ via (4.10). Similarly, Y and the ρ_j determine $\mathcal{F}|_{\zeta^2=\infty}$.

Remark 4.9. Symplectic leaves of $\mathcal{M}(k,l)^0/K$ can be also identified with reduced orbits (cf. Definition 4.5) of $\widetilde{GL}(l)^+$ in $\widetilde{\mathfrak{gl}}(l)^-$. Therefore, the last proposition describes sheaves corresponding to a reduced orbit with Y semisimple. Furthermore, if we view $\mathcal{M}(k,l)^0/K$ as an open subset of the moduli space of semistable sheaves with Hilbert polynomial $lx + ky$, then this map is a symplectomorphism between the Mukai-Tyurin-Bottacin symplectic structure, described in the introduction, and the Kostant-Kirillov form on a reduced orbit of a Lie group. For an open dense set, where $\mathcal{F}$ is a line bundle on a smooth curve, this follows from results in [2, 4]. Since both symplectic structures extend everywhere, they are must be isomorphic everywhere.

Example 4.10. If we want $\mathcal{F}$ to be a line bundle over its support, then we must require that all k_i and all l_j are equal to 1. A symplectic leaf in $\mathcal{M}(k,l)^0/K$ is now given by fixing diagonal matrices $X = \mathrm{diag}(\zeta_1, \ldots, \zeta_k)$ and $Y = \mathrm{diag}(\eta_1, \ldots, \eta_l)$ with all ζ_i and all η_j distinct, as well as the diagonal entries of FG and GF, and quotienting by the group of $(k+l) \times (k+l)$ diagonal matrices (acting as in (4.3)). If the diagonal entries of FG are fixed to be $\alpha_1, \ldots, \alpha_k$, and the diagonal entries of GF are $\beta_1, \ldots, \beta_l$, then the corresponding subset of $\mathcal{S}(k,l)$ consists of sheaves $\mathcal{F}$ supported on a 1-dimensional scheme S such that

$$S \cap \{\eta^2 = \infty\} = \bigcup_{i=1}^k \left\{\zeta - \zeta_i = \frac{\alpha_i}{\eta}\right\}, \quad S \cap \{\zeta^2 = \infty\} = \bigcup_{j=1}^l \left\{\eta - \eta_j = \frac{\beta_j}{\zeta}\right\}$$

and the rank of $\mathcal{F}$ restricted to $S \cap \{\eta^2 = \infty\}$ and $S \cap \{\eta^2 = \infty\}$ is everywhere 1.

Remark 4.11. We expect that Proposition 4.8 remains true if X or Y are not semisimple.

5. Rank k perturbations

Let us now assume that $k \leq l$. In [1], the authors consider Hamiltonian flows on a subset $\mathcal{M}$ of $\mathcal{M}^0(k,l)/K$, where $\operatorname{rank} F = \operatorname{rank} G = k$. It is clear from the

previous section that a generic symplectic leaf of $\mathcal{M}^0(k,l)/K$ is not contained in $\mathcal{M}$. Therefore a flow may leave $\mathcal{M}$ without becoming singular. Since such Hamiltonian flows on a particular symplectic leaf can be linearised on the Jacobian of a spectral curve, it is interesting to know which points of the (affine) Jacobian are outside of $\mathcal{M}$. We are going to give a very satisfactory answer to this, in terms of cohomology of line bundles.

Let us therefore define the following set:

$$\mathcal{M}(k,l)^1 = \{M \in \mathcal{M}(k,l)\,;\ \operatorname{rank} F = \operatorname{rank} G = k\}\,. \tag{5.1}$$

Remark 5.1. The manifold of $GL_k(\mathbb{C})$-orbits in $\mathcal{M}(k,l)^1$ with $X=0$ and fixed Y, can be identified with the set $\{Y+GF\}$, i.e. with the *space of rank k perturbations of the matrix* Y, as considered first by Moser [13] ($k=2$), and, then by many other authors, in particular Adams, Harnad, Hurtubise, Previato [5, 1].

We now ask which acyclic sheaves on $\mathbb{P}^1\times\mathbb{P}^1$ correspond to orbits of $K = GL_k(\mathbb{C})\times GL_l(\mathbb{C})$ on $\mathcal{M}(k,l)^1$. We have:

Proposition 5.2. *Let $k\le l$. The bijection of Corollary 2.6 induces a bijection between:*

(i) *orbits of $GL_k(\mathbb{C})\times GL_l(\mathbb{C})$ on $\mathcal{M}(k,l)^1$, and*
(ii) *isomorphism classes of acyclic sheaves $\mathcal{F}$ on $\mathbb{P}^1\times\mathbb{P}^1$ with Hilbert polynomial $P_{\mathcal{F}}(x,y)=lx+ky$, which satisfy, in addition,* (4.1) *and*

$$H^0(\mathcal{F}(-1,1))=0 \ \text{ and } \ H^1(\mathcal{F}(1,-1))=0.$$

Proof. Consider short exact sequences

$$0\to\mathcal{O}(-1)^{\oplus k}\xrightarrow{(X-\zeta,G)^T}\mathcal{O}^{\oplus(k+l)}\longrightarrow\mathcal{W}_1\to 0,$$

$$0\to\mathcal{O}(-1)^{\oplus l}\xrightarrow{(F,Y-\eta)^T}\mathcal{O}^{\oplus(k+l)}\longrightarrow\mathcal{W}_2\to 0.$$

The condition that G has rank k is equivalent to $\mathcal{W}_1$ being a vector bundle, isomorphic to $\mathcal{O}(1)^{\oplus k}\oplus\mathcal{O}^{\oplus(l-k)}$. This is equivalent to $H^0(\mathcal{W}_1\otimes\mathcal{O}(-2))=0$. On the other hand, we claim that the condition that F has rank k is equivalent to $H^1(\mathcal{W}_2\otimes\mathcal{O}(-2))=0$. Indeed, any coherent sheaf on $\mathbb{P}^1$ splits into sum of line bundles $\mathcal{O}(i)$ and a torsion sheaf [16]. Since $\mathcal{W}_2$ has a resolution as above, we know that all degrees i in the splitting are nonnegative, and F has rank k if and only if all i are strictly positive, which is equivalent to $H^1(\mathcal{W}_2\otimes\mathcal{O}(-2))=0$.

We can use the above exact sequences to obtain two further resolutions of $\mathcal{E}=\mathcal{F}(1,1)$:

$$0\to\mathcal{O}(-1,0)^{\oplus k}\to\pi_2^*\mathcal{W}_2\to\mathcal{E}\to 0, \tag{5.2}$$

$$0\to\mathcal{O}(0,-1)^{\oplus l}\to\pi_1^*\mathcal{W}_1\to\mathcal{E}\to 0, \tag{5.3}$$

where the maps between first two terms are given by the embedding in $\mathcal{O}^{\oplus(k+l)}$ followed by the projection onto the quotients $\mathcal{W}_2$, $\mathcal{W}_1$. Tensoring (5.2) with $\mathcal{O}(0,-2)$ shows that $H^1(\mathcal{W}_2(-2))=0$ if and only if $H^1(\mathcal{E}(0,-2))=0$, i.e. $H^1(\mathcal{F}(1,-1)=0$. Similarly, tensoring (5.3) with $\mathcal{O}(-2,0)$ shows that $H^0(\mathcal{W}_1(-2))=0$ if and only if $H^0(\mathcal{E}(-2,0))=0$, i.e. $H^0(\mathcal{F}(-1,1)=0$. □

Remark 5.3. In the case $k = l$, $H^0(\mathcal{E}(-2,0)) = 0$ implies that $\mathcal{E}(-2,0)$ is acyclic (and similarly, $H^1(\mathcal{E}(0,-2)) = 0$ implies that $\mathcal{E}(0,-2)$ is acyclic). In other words $\mathcal{G} = \mathcal{E}(-1,0)$ satisfies $H^*(\mathcal{G}(-1,0)) = H^*(\mathcal{G}(0,-1)) = 0$. Furthermore, the resolution (5.3) becomes the following resolution of $\mathcal{G}$:

$$0 \to \mathcal{O}(-1,-1)^{\oplus k} \longrightarrow \mathcal{O}^k \longrightarrow \mathcal{G} \to 0. \tag{5.4}$$

In the case when $S = \operatorname{supp} \mathcal{G}$ is smooth and $\mathcal{G}$ is a line bundle, the corresponding part of $\mathrm{Jac}^{g+k-1}(S)$ and the resolution (5.4) have been considered by Murray and Singer in [15].

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